

Covering spaces and the Kakimizu complex

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Abstract

We consider how the universal abelian cover of a knot exterior sheds light on the Kakimizu complex of the knot. First, we introduce the notion of covering spread for pairs of Seifert surfaces of a knot and prove that it is equal to the distance less 1 of the corresponding vertices in the Kakimizu complex. This equivalence provides an effective means of computing distance in the Kakimizu complex. Second, we prove that the Kakimizu complex is simply connected.

One of the fundamental objects considered in the topological study of knots is the Seifert surface. Interestingly, a knot can have many, and in some cases infinitely many, disjoint Seifert surfaces. The Kakimizu complex aims to capture structural information of the set of all Seifert surfaces of a given knot. It is one of several complexes defined by considering isotopy classes of certain submanifolds and disjointness properties of representatives of such isotopy classes.

One of the first to study this complex was O. Kakimizu (see for instance [5]), though the discovery of some facts concerning this complex date further back. For instance, the results of M. Scharlemann and A. Thompson in [11] establish that the Kakimizu complex is connected.

Recent years have seen progress in understanding key facts about the Kakimizu complex: W. Jaco and E. Sedgwick showed that the Kakimizu complex of the knot K is finite if K is atoroidal and has genus at least 2 (see [7]). Moreover, R. Wilson showed that, in fact, it suffices to assume that K is atoroidal (see [13]). Results pertaining to specific classes of knots can be found in [2], [9] and [12]. In [10], M. Sakuma and K. Shackleton establish concrete diameter bounds and provide an overview of the current understanding of the Kakimizu complex. In particular, they prove that the Kakimizu complex is simply connected for knots of genus 1. A more general understanding of the shape of the Kakimizu complex is highly desirable. Many questions remain unanswered. Though we establish simple connectivity here, the conjectured contractibility has yet to be proved.

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1 Preliminaries

Here we state key definitions and results. For basic definitions concerning knots, see [1], [6], or [8].

Definition 1. Let K be a knot in \mathbb{S}^3 . Denote an open regular neighborhood of K by $\eta(K)$ and the exterior of K , $\mathbb{S}^3 - \eta(K)$, by $E(K)$.

Definition 2. The Seifert surfaces of a knot K form a simplicial complex as follows: 1) Vertices correspond to isotopy classes of minimal genus Seifert surfaces; 2) Edges correspond to pairs of disjoint Seifert surfaces and, more generally, n -dimensional faces correspond to $(n + 1)$ -tuples of pairwise disjoint Seifert surfaces.

This complex is called the Kakimizu complex.

Definition 3. The link of a simplex σ in a simplicial complex X , denoted by X_σ , is the union of all simplices disjoint from σ , that together with σ span a simplex in X . The residue of a simplex σ in a simplicial complex X , denoted by $\text{Res}(X, \sigma)$, is the union of all simplices in X that contain σ .

In our discussion here, paths and cycles in a simplicial complex will traverse only vertices and edges (not higher dimensional faces).

Definition 4. The distance between two vertices v, v^* in the Kakimizu complex, denoted by $d_K(v, v^*)$ is the minimal possible number of edges in a path connecting the two vertices. The length of a cycle is the number of edges in the cycle.

Definition 5. A cycle is said to be trivial if it is homotopic to a point. Otherwise, it is nontrivial.

Definition 6. A simplicial complex X is k -large if every nontrivial cycle in X has length at least k and for every simplex σ of X , every nontrivial cycle in X_σ has length at least k .

Definition 7. A simplicial complex is locally k -large if the residue of every simplex in X is k -large.

Theorem 1. (Januszkiewicz-Swiątkowski) The universal cover of a finite dimensional connected locally 6-large simplicial complex is contractible.

Below, we will apply Theorem 1 in the special case in which the Kakimizu complex is 2-dimensional. In this case, the above result is classical.

Definition 8. Let S, S^* be two Seifert surfaces of K . Denote the number of components of $S \cap S^*$, called the intersection number of S and S^* , by $i(S, S^*)$.

One of the fundamental results concerning the Kakimizu complex is due to M. Scharlemann and A. Thompson. In the language here, it can be formulated as follows:

Theorem 2. (*Scharlemann-Thompson*) *The Kakimizu complex is connected. Moreover, given two Seifert surfaces S, S^* , the distance of the corresponding vertices in the Kakimizu complex is bounded above by $i(S, S^*) + 1$.*

We wish to examine the role of the universal abelian cover of the knot complement in shedding light on distances in the Kakimizu complex. We do so by lifting distinct Seifert surfaces to the universal abelian cover and considering their parallel translates. It is then possible to partition the curves of intersection between two Seifert surfaces according to the parallel translates of lifts in which they occur. We do so in the proof of Lemma 3 below. The framework relies on the concept of “covering spread”.

Definition 9. *Let S, S^* be two Seifert surfaces of K , considered to be lying in $E(K)$. Denote the universal abelian cover of $E(K)$ by $M(K)$. Let τ be a generator of the group of covering translations of $M(K)$. Isotope S and S^* so that ∂S and ∂S^* are disjoint (and parallel) curves in $\partial E(K)$ and the number of components of $S \cap S^*$ is minimal. Choose a lift S_0 , in $M(K)$, of S . Denote the translation of S_0 under τ^n , for $n \in \mathbb{Z}$, by S_n .*

Denote the lift of S^ with boundary between S_0 and S_1 by S_0^* and denote the translation of S_0^* under τ^n , for $n \in \mathbb{N}$, by S_n^* . Set*

$$l_t = \max\{n \in \mathbb{N} \mid S_n \cap S_0^* \neq \emptyset\} \text{ if } S_1 \cap S_0^* \neq \emptyset$$

and 0 if $S_1 \cap S_0^ = \emptyset$. Set*

$$l_b = \max\{n \in -\mathbb{N} \mid S_n \cap S_0^* = \emptyset\} \text{ if } S_0 \cap S_0^* \neq \emptyset$$

and 0 if $S_0 \cap S_0^ = \emptyset$. The covering spread of S and S^* , denoted by $cs(S, S^*)$, is the difference $l_t - l_b$. See Figure 1.*

Remark 10. *It follows from the definitions that*

$$cs(S, S^*) \leq i(S, S^*)$$

A key tool used by M. Scharlemann and A. Thompson is the double curve sum of two surfaces along a curve.

Definition 11. *Let P and Q be oriented surfaces in an orientable 3-manifold and let c be a subset of $P \cap Q$ such that any component of $P \cap Q$ is either disjoint from or entirely contained in c . Denote a closed regular neighborhood of c by $N(c)$. Here each component of $N(c)$ meets $P \cup Q$ in two annuli, $A_p \subset P$ and $A_q \subset Q$, that intersect in a component of c . We can remove the interiors of A_p and A_q and cap off the resulting boundary components with two components of $\partial N(c) \setminus (\partial(A_p \cup A_q))$. Moreover, we may do so in such a way that the orientations of the remnants of P and Q match up. The double curve sum of P and Q along c is the result of performing such replacements along all components of c . See Figure 2.*

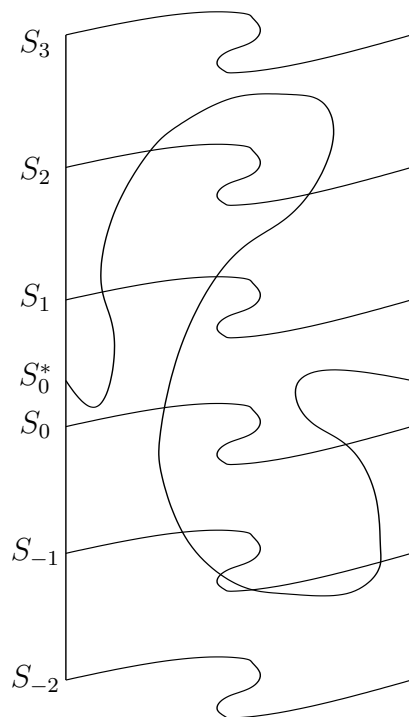


Figure 1: *Lifts of S and a lift of S^**



Figure 2: *A cross section of the double curve sum*

A fundamental result in the paper of M. Scharlemann and A. Thompson is the following:

Lemma 1. *(Scharlemann-Thompson) Let S, S^* be two minimal genus Seifert surfaces of K (considered in $E(K)$), isotoped so as to intersect in a minimal number of components. Then the double curve sum of S and S^* along $S \cap S^*$ contains two minimal genus Seifert surfaces.*

The following lemma is immediate, yet it is essential for our purposes here.

Lemma 2. *The projection of the double curve sum of S_k and S_0^* along $S_k \cap S_0^*$ is equivalent to the double curve sum of S and S^* along the projection of $S_k \cap S_0^*$.*

Proof: Here the replacements for the double curve sum all take place in a small neighborhood of S_n . The lemma now follows from the fact that there is a homeomorphism between this neighborhood of S_n and a neighborhood of S in $E(K)$. \square

2 Covering spread and distance

Like Scharlemann and Thompson, we wish to use the double curve sum to create surfaces that interpolate between S and S^* . But, rather than taking a double curve sum of S and S^* along all components of $S \cap S^*$ we do so along a carefully chosen subcollection. This is done by employing the notion of covering spread. A first step is to find a surface disjoint from S^* that has lower covering spread with S . To this end, we work with the universal abelian cover $M(K)$ of $E(K)$, as above.

The techniques and results from PL-minimal surface theory prove useful in this context. (See [4] for details and for standard results on PL-minimality.) Most important here are two facts about PL-minimal surfaces: 1) Two PL-minimal surfaces that represent isotopy classes of surfaces with disjoint representatives are either disjoint or equal; 2) Lifts of PL-minimal surfaces are PL-minimal.

The following lemma uses the notation from Definition 9.

Lemma 3. *Let S, S^* be minimal genus Seifert surfaces of K (considered in $E(K)$), isotoped so as to intersect in a minimal number of components. Suppose that*

$$cs(S, S^*) > 1.$$

Take the double curve sum of S_{t_t} and S_0^ along $S_{t_t} \cap S_0^*$. This double curve sum contains two components, S_0^{*-} and S_0^{*+} , with nonempty boundary. Denote the projection of S_0^{*-} by S^{*-} . After a small isotopy, S^{*-} is a minimal genus Seifert surface that is disjoint from S^* and $cs(S, S^{*-}) < cs(S, S^*)$.*

Proof: Let \mathcal{T} be a triangulation of $E(K)$ and suppose that S, S^* are PL-minimal. Here S_0^* is separating in $M(K)$. Denote the two components of $M(K) \setminus S_0^*$ by M_0^{*-} and M_0^{*+} , with M_0^{*+} the component above S_0^* , i.e., that containing S_1^* . Similarly,

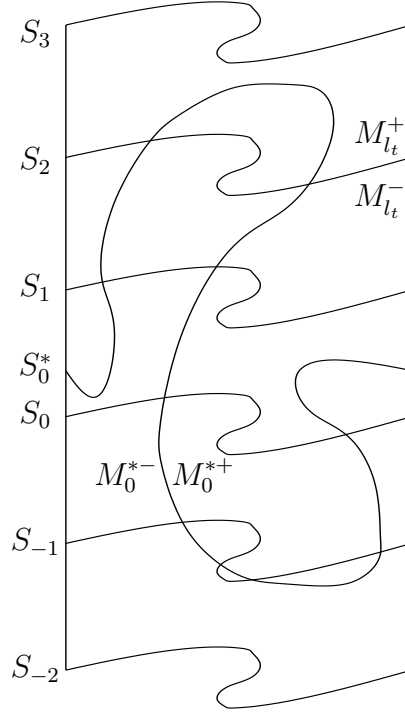


Figure 3: Labellings M_0^{*-} , M_0^{*+} , $M_{l_t}^+$ and $M_{l_t}^-$



Figure 4: A cross section of the double curve sum near $S_{l_t} \cap S_0^*$

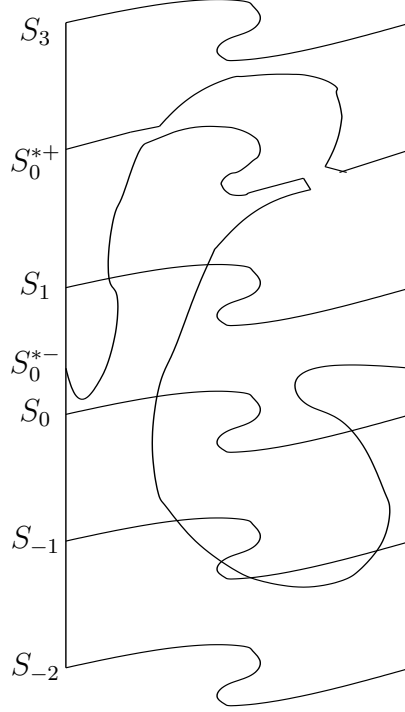


Figure 5: *The double curve sum of S_0^* and S_{l_t} along their intersection*

denote the two components of $M(K) \setminus S_{l_t}$ by $M_{l_t}^-$ and $M_{l_t}^+$ with $M_{l_t}^+$ the component above S_{l_t} , *i.e.*, that containing S_{l_t+1} . See Figure 3.

Note that $S_{l_t} \cap S_0^*$ separates S_{l_t} into the two nonempty subsurfaces $S_{l_t} \cap M_0^{*\pm}$ and S_0^* into the two nonempty subsurfaces $S_0^* \cap M_{l_t}^\pm$. Specifically, $S_{l_t} \cap S_0^*$ is separating in both S_{l_t} and S_0^* . Take the double curve sum of S_{l_t} and S_0^* along $S_{l_t} \cap S_0^*$. The result is partitioned into two subsurfaces: One subsurface isotopic to the boundary of $M_{l_t}^+ \cap M_0^{*+}$ and a complementary subsurface isotopic to the boundary of $M_{l_t}^- \cap M_0^{*-}$. See Figures 4 and 5. (In the case $l_t > 0$, the boundary of the former contains ∂S_{l_t} and can't contain ∂S_0^* , whereas the boundary of the latter contains ∂S_0^* and can't contain ∂S_{l_t} . In the case $l_t = 0$, the situation is reversed.)

The two subsurfaces discussed above need not be connected, but each contains exactly one boundary component. Denote the component with nonempty boundary of the subsurface isotopic to the boundary of $M_{l_t}^- \cap M_0^{*-}$ by S_0^{*-} and the component with nonempty boundary of the subsurface isotopic to the boundary of $M_{l_t}^+ \cap M_{*,0}^+$ by S_0^{*+} .

Since the surface S_0^{*-} is isotopic to a component of the boundary of $M_{l_t}^- \cap M_0^{*-}$, it can be made disjoint from S_0^* and from S_{l_t} by isotoping it off of $\partial(M_{l_t}^- \cap M_{*,0}^-)$ into $M_{l_t}^- \cap M_0^{*-}$. In particular, S_0^{*-} lies below S_0^* after this isotopy. Furthermore, for

$k < 0$, S_k^* does not meet S_{l_t} and thus also does not meet a regular neighborhood of S_{l_t} . In particular, S_0^{*-} lies above S_{-1}^* . Thus S_0^{*-} is an embedded surface that lies in the interior of the fundamental domain of $M(K)$ bounded by S_1^* and S_0^* . It follows directly that the projection, S^{*-} , of S_0^{*-} is an embedded surface that is disjoint from S^* . Furthermore, the boundary of S^{*-} is a longitude of $\partial E(K)$, hence S^{*-} is a Seifert surface.

By construction, the surface S^{*+} lies in the fundamental region between S_{l_t} and S_{l_t+1} , thus it too projects to an embedded surface. The boundary of S^{*+} is also a longitude of $\partial E(K)$, hence S^{*+} is also a Seifert surface. By Lemma 2, S^{*-} and S^{*+} are contained in the double curve sum of S and S^* along a subset of their curves of intersection. (Note that S^{*-} and S^{*+} need not be disjoint.) Thus

$$\chi(S) + \chi(S^*) \leq \chi(S^{*-}) + \chi(S^{*+})$$

It follows that S^{*-} and S^{*+} are minimal genus Seifert surfaces. \square

The projection of $S_0^* \cap S_{l_t}$ singles out components of $S^* \cap S$. A posteriori, we see that we can take the double curve sum along only these curves of intersection to produce a Seifert surface interpolating between S^* and S , rather than along all of $S^* \cap S$, as was done in the proof of Theorem 2.

Lemma 4. *Suppose that*

$$cs(S, S^*) = 1$$

and take the double curve sum of S_{l_t} and S_0^ to produce S^{*-} and S^{*+} as above. Then S^{*-} and S^{*+} satisfy $S^{\pm} \cap S = \emptyset$ and $S^{*\pm} \cap S^* = \emptyset$.*

Proof: Let \mathcal{T} be a triangulation of $E(K)$ and suppose that S, S^* are PL-minimal. Under the assumptions here, there are two possibilities: 1) In the case that $l_t = 1$ and $l_b = 0$, S_0^{*+} lies strictly between S_1 and S_2 and also strictly between S_0^* and S_1^* . Likewise, S_0^{*-} lies strictly between S_0 and S_1 and also strictly between S_{-1}^* and S_0^* . So the conclusion follows. See the schematic in Figure 6. 2) In the case that $l_t = 0$ and $l_b = -1$, S_0^{*+} lies strictly between S_0 and S_1 and also strictly between S_0^* and S_1^* . Likewise, S_0^{*-} lies strictly between S_{-1} and S_0 and also strictly between S_{-1}^* and S_0^* . So the conclusion follows. \square

We will use Lemma 4 in the proof of Theorem 5.

It is tempting to try to employ this idea in the context of other complexes. But note that, for instance in the curve complex, this construction leads nowhere, because the separating properties do not carry over. Specifically, some closed curves lift to lines rather than closed curves and thus have infinite covering spread.

Theorem 3. *Let S, S^* be two minimal genus Seifert surfaces of the knot K and let v, v^* be the corresponding vertices of the Kakimizu complex. Then*

$$d_K(v, v^*) = cs(S, S^*) + 1$$

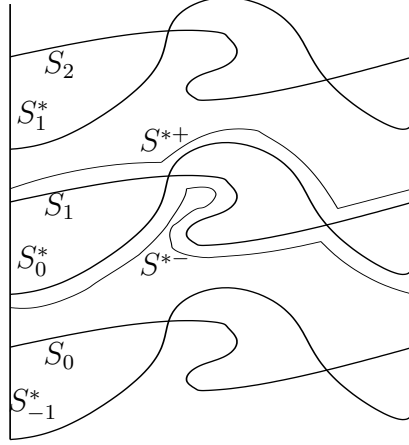


Figure 6: *The double curve sum of S_0^* and S_1 along their intersection*

Proof: We first show that

$$d_K(v, v^*) \leq cs(S, S^*) + 1$$

In the case $cs(S, S^*) = 0$, S and S^* are disjoint and hence $d_K(v, v^*) = 1$. In the case $cs(S, S^*) > 0$, Lemma 3 furnishes the surface S^{*-} that is disjoint from S^* . Thus for v^{*-} the vertex corresponding to S^{*-} , $d_K(v^{*-}, v^*) = 1$. Furthermore, by construction, $cs(S, S^{*-}) \leq cs(S, S^*) - 1$. The inequality now follows by induction.

Next we show that

$$d_K(v, v^*) \geq cs(S, S^*) + 1$$

If $d_K(v, v^*) = 1$, then S and S^* are disjoint and hence $cs(S, S^*) = 0$. Now suppose that the inequality is true whenever $d_K(v, x) \leq n$ and that $d_K(v, v^*) = n + 1$. Let $v^1 = v, v^2, \dots, v^n, v^{n+1} = v^*$ be a path in the Kakimizu complex connecting v and v^* and let $S^1 = S, S^2, \dots, S^n, S^{n+1} = S^*$ be representatives for the vertices in this path.

Let \mathcal{T} be a triangulation of $E(K)$ and suppose that S^1, \dots, S^{n+1} are PL-minimal. By the above, $cs(S^n, S^*) = 0$. Furthermore, by the inductive hypothesis, $cs(S, S^n) + 1 \leq d_K(v, v^n)$, so $cs(S, S^n) \leq n - 1$. Consider the fundamental regions between lifts of S^n . Each such fundamental region contains exactly one lift of S^* . The number of such regions met by a lift of S is at most $cs(S, S^n) + 1 \leq n$. It follows that S meets at most $cs(S, S^n) + 1 \leq n$ distinct lifts of S^* . Whence $cs(S, S^*) \leq n + 1$. See Figure 7. \square

Corollary 4. *There is an algorithm to compute distances in the Kakimizu complex.*

Proof: Given two vertices v, v' in the Kakimizu complex, choose PL-minimal representatives S, S' . Lift S, S' to $M(K)$ and compute the covering spread. \square

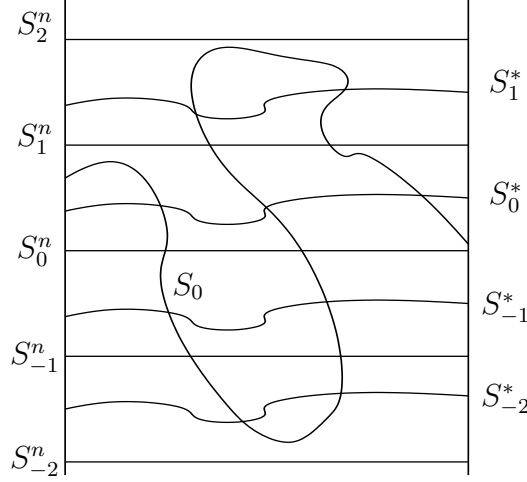


Figure 7: *The fundamental regions defined by S^n*

3 The Kakimizu complex is simply connected

We here prove that the Kakimizu complex is simply connected. The proof relies on the universal abelian cover only in the context of applying Lemma 4. But this step in the proof is essential! The argument is loosely inspired by an earlier incorrect argument.

Theorem 5. *The Kakimizu complex is simply connected.*

Proof: Let $K \subset \mathbb{S}^3$ be a knot and let v^1, \dots, v^n be the vertices in a cycle in the Kakimizu complex of K . Let S^1, \dots, S^n be representatives of v^1, \dots, v^n . Abusing notation slightly, we write $1, \dots, n$ when we really mean $1 \bmod n, \dots, n \bmod n$. Let \mathcal{T} be a triangulation of $E(K)$ and suppose that S^1, \dots, S^n are PL-minimal.

The complexities of PL surfaces are ordered pairs considered in the dictionary topology. Suppose now that

$$c(S^i) \geq c(S^{i-1}), c(S^{i+1})$$

If $S^{i-1} \cap S^{i+1} = \emptyset$, then either v^1, \dots, v^n is a cycle of length three and hence spans a 2-simplex in the Kakimizu complex of K , or there is a shorter cycle, homotopic to the original cycle, obtained by replacing v^{i-1}, v^i, v^{i+1} in the original cycle with v^{i-1}, v^{i+1} .

If $S^{i-1} \cap S^{i+1} \neq \emptyset$, then the distance of the corresponding vertices in the Kakimizu complex is 2. Hence by Theorem 3, $cs(S^{i-1}, S^{i+1}) = 1$. Thus by Lemma 4, the double curve sum of these two surfaces yields two new Seifert surfaces, $(S^i)^+$ and $(S^i)^-$, such that both $(S^i)^\pm \cap S^{i-1} = \emptyset$ and $(S^i)^\pm \cap S^{i+1} = \emptyset$. Here $S^{i\pm 1}$ are disjoint from S^i and $(S^i)^\pm$ lie in a regular neighborhood of $S^{i-1} \cup S^{i+1}$, hence $(S^i)^\pm \cap S^i = \emptyset$.

By the exchange and roundoff trick and the Meeks-Yao trick

$$c((S^i)^+) + c((S^i)^-) < c(S^{i-1}) + c(S^{i+1})$$

Take $(S^i)'$ to be that of $(S^i)^+$ or $(S^i)^-$ that has complexity no larger than the other. Then $c((S^i)') < c(S^i)$ and we form a new cycle $v^1, \dots, v^{i-1}, (v^i)', v^{i+1}, \dots, v^n$, with $(v^i)'$ the vertex in the Kakimizu complex of K represented by $(S^i)'$. Because

$$(S^i)' \cap S^i = \emptyset,$$

$$(S^i)' \cap S^{i\pm 1} = \emptyset,$$

$$S^i \cap S^{i\pm 1} = \emptyset,$$

the new cycle is homotopic, in the Kakimizu complex, to our original cycle.

Recall that the number of PL-minimal surfaces of a bounded complexity, and hence the number of n -tuples of PL-minimal surfaces of a bounded complexity, is finite. Thus if we suppose that there is a cycle in the Kakimizu complex that is non trivial, then we may assume that the (non trivial) cycle v^1, \dots, v^n and the representatives S^1, \dots, S^n are chosen so that the sum of the complexities of S^1, \dots, S^n is minimal among all cycles in the Kakimizu complex of K that are not homotopically trivial. It follows from the preceeding paragraph and from the fact that there are no homotopically non trivial cycles of length three that then

$$c(S^i) < c(S^{i-1}) + c(S^{i+1})$$

for all S^i . But recall that here i means $i \bmod n$ and $i \pm 1$ means $i \pm 1 \bmod n$. So this is impossible. \square

The schematic diagram in Figure 8 shows how a cycle in the Kakimizu complex is successively “filled in” by the constructions in the proof of Theorem 5.

The phenomenon here may be more general: For instance, it might apply to a complex defined analogously to the Kakimizu complex on the Thurston norm minimizing representatives of a primitive homology class of an orientable compact 2-manifold or 3-manifold.

The following lemma could be established via an argument analogous to that in Theorem 5. Yet the phenomenon appears to be more general. For this reason we include the more basic argument below. The only property used here that is specific to the Kakimizu complex is the fact that cycles of length 3 cobound 2-simplices.

Lemma 5. *The residues of simplices in the Kakimizu complex of a knot are simply connected.*

Proof: Let $\sigma = (x_1 \dots x_n)$ be a simplex in the Kakimizu complex and suppose that c is a cycle in the residue of σ . If no vertex of c is contained in σ , then each edge in c spans a simplex together with, say, the vertex x_1 . Thus c is homotopically trivial in the residue of σ .

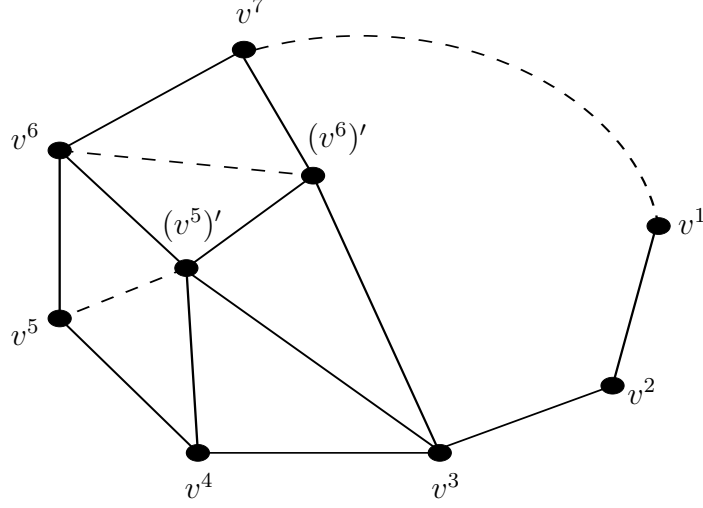


Figure 8: A cycle in the Kakimizu complex

Suppose that at least one vertex in c lies in σ and at least one closed edge $e = (v_2v_3)$ of c does not. Then we may assume, by rechoosing e , if necessary, that the edge $e' = (v_1v_2)$ preceding (v_2, v_3) in the cycle c has $v_1 \in \sigma$. Here e lies in the residue of σ , hence there is a simplex τ that contains $e \cup \sigma$. It follows that τ contains the simplex $(v_1v_2v_3)$ and hence that c is homotopic, in the residue of σ , to a cycle \hat{c} obtained from c by replacing the edges $e' \cup e$ with the edge $e'' = (v_1v_3)$.

If exactly one of the vertices, v , of c does not lie in σ , then the residue of σ must contain a simplex, τ' , containing both v and σ . Hence c lies in τ' , where it is homotopically and thus it is homotopically trivial in the residue of σ . If c lies entirely in σ , then it is homotopically trivial in σ , hence in the residue of σ . It follows by induction that c is homotopically trivial in the residue of σ . \square

Lemma 6. *A nontrivial cycle in the link of a vertex in the Kakimizu complex must have length at least 6.*

Proof: To prove that such a cycle must have length at least 5, we argue as in the proof of Theorem 5. But here there are subtle differences. Specifically, let v be a vertex and suppose the cycle v^1, v^2, v^3, v^4 lies in X_v , the link of v . Following the argument, and the notation, in the proof of Theorem 5, we must here consider the case in which one or both of the surfaces $(S^i)^\pm$ are isotopic to S , the PL-minimal representative of v . (Recall that S does not correspond to a vertex in X_v .)

If at least one, call it $(S^i)''$, of $(S^i)^\pm$ is not isotopic to S , then it may have larger complexity than $S^{i\pm 1}$. (So the minimality argument in the proof of Theorem 5 yields no conclusion.) Fortunately, the situation here is more constrained: $S_0^1, S_0^2, S_0^3, S_0^4$ lie between S_0 and S_1 . Moreover,

$$S_0^1 \cap S_0^2 = S_0^1 \cap S_0^4 = S_0^3 \cap S_0^4 = S_0^3 \cap S_0^2 = \emptyset$$

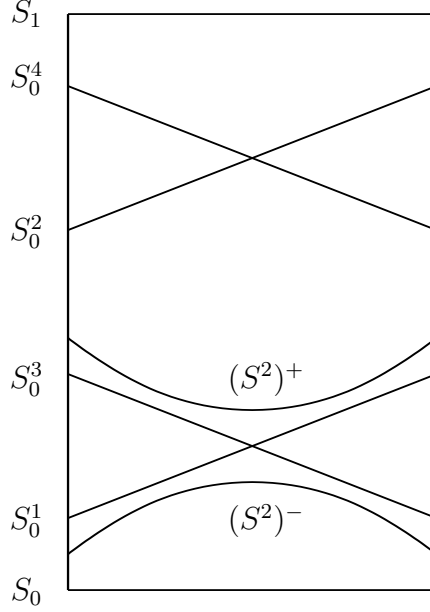


Figure 9: *Lifts of representatives of a cycle in X_v*

and by construction,

$$(S^i)'' \cap S = \emptyset$$

Furthermore, we may assume that

$$S_0^1 \cap S_0^3 \neq \emptyset$$

and

$$S_0^2 \cap S_0^4 \neq \emptyset$$

for otherwise the cycle breaks down into two 3-cycles, each spanning a 2-simplex, and is hence trivial. See Figure 9.

In this setting, $(S^i)''$ is a surface that results from either the double curve sum of S^1 and S^3 or from the double curve sum of S^2 and S^4 . In the first case it is disjoint from S^2 and S^4 , as these lie outside a regular neighborhood of $S^1 \cup S^3$. In the second case it is disjoint from S^1 and S^3 , as these lie outside a regular neighborhood of $S^2 \cup S^4$. Thus the corresponding vertex, $(v^i)''$, in the Kakimizu complex is distance 1 from v^1, v^2, v^3 and v^4 and hence spans a 2-simplex together with each edge in the cycle v^1, v^2, v^3, v^4 . Since $(S^i)''$ is also disjoint from S , $(v^i)''$ has distance 1 from v , in particular, $(v^i)'' \in X_v$. This implies that the cycle is trivial in X_v . See Figure 10.

If both surfaces $(S^i)^{\pm 1}$ are isotopic to S , we must argue differently. In this case $E(K) \setminus S_i^{\pm 1}$ has two components one of which must be a product homeomorphic to $S \times I$. In particular, all Seifert surfaces contained in the product must be isotopic to S . This means that either the two surfaces $S^{i\pm 1}$ are isotopic to S or the remaining

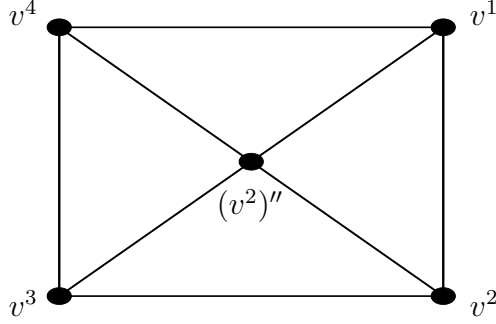


Figure 10: A cycle of length 4 in X_v is trivial in X_v

two surfaces among S^1, S^2, S^3, S^4 are isotopic to S . But this is impossible. Thus there are no homotopically nontrivial cycles of length less than or equal to 4 in the link of v .

Now suppose that v^1, \dots, v^5 is a cycle of length 5 in X_v and let S^1, \dots, S^5 be PL-minimal representatives of v^1, \dots, v^5 . Consider the surfaces S_0^1, \dots, S_0^5 in $M(K)$. They lie between S_0 and S_1 . If

$$S^i \cap S^j = \emptyset \text{ for } j \neq i, i \pm 1 \pmod{5},$$

then c is homotopic, in X_v , to a cycle of length 4. By the above, this cycle is trivial in X_v . So we may assume that

$$S^i \cap S^j \neq \emptyset \text{ for } j \neq i, i \pm 1 \pmod{5}$$

In particular, $S_0^2 \cap S_0^5 \neq \emptyset$, so the two surfaces must either both lie above or both lie below S_0^1 . Assume the former, as the other case will then follow by symmetry. Also, S_0^3 must lie below S_0^2 , in order to have nonempty intersection with S_0^1 . Furthermore, S_0^4 must lie above S_0^3 in order to have nontrivial intersection with S_0^2 , but below S_0^5 , in order to have nontrivial intersection with S_0^1 . Since $S_0^3 \cap S_0^5 \neq \emptyset$, this is impossible. \square

This lemma may or may not hold for links of higher dimensional simplices. Specifically, in the case of the link of an edge, one must consider not only surfaces isotopic to S , but also another surface, S' . If one of S_i^\pm is isotopic to S and the other is isotopic to S' , the argument used here breaks down.

Theorem 6. *If the Kakimizu complex of a knot K is at most 2-dimensional, then it is contractible.*

Proof: If the Kakimizu complex is 2-dimensional, then the links of 2-simplices are empty and the links of 1-simplices are 0-dimensional. Lemma 5 establishes that residues are simply connected. Lemma 6 establishes the fact that links of vertices in the Kakimizu complex are simply connected. These two facts combine to ensure

that residues are 6-large. Thus the hypotheses of Theorem 1 are satisfied, whence the universal cover of the Kakimizu complex is contractible. Hence by Theorem 5 the Kakimizu complex is contractible. \square

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